

THE IRRATIONALITY OF A NUMBER THEORETICAL SERIES

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ABSTRACT. Denote by $\sigma_k(n)$ the sum of the k -th powers of the divisors of n , and let $S_k = \sum_{n \geq 1} \frac{\sigma_k(n)}{n!}$. We prove that Schinzel's conjecture H implies that S_k is irrational, and give an unconditional proof for the case $k = 3$.

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Let $\sigma_k(n) = \sum_{d|n} d^k$, and set $S_k = \sum_{n \geq 1} \frac{\sigma_k(n)}{n!}$. For $k = 0, 1$ it follows from a general result by Erdős and Straus [3], that S_k is irrational, whereas for $k = 2$ the same was shown by Erdős and Kac[2]. In [1], Erdős posed the question whether S_k is irrational for all k . We will prove the following theorem.

Theorem. *Define S_k as above.*

- (1) *If Schinzel's conjecture H is true, then S_k is irrational for all $k \in \mathbb{N}$.*
- (2) *S_3 is irrational.*

Here, Schinzel's conjecture H is the following generalization of the prime twin conjecture (cf. [8]):

Let P_1, \dots, P_k be integral polynomials with positive leading coefficients, such that for each prime number p there exists some integer a such that $P_1(a) \cdots P_k(a) \not\equiv 0 \pmod{p}$. Then there exist infinitely many integers n such that $P_i(n)$ is prime for $1 \leq i \leq k$.

Proof. Assume that S_k was rational, say, $S_k = \frac{a}{b}$, $(a, b) = 1$. Then for every $n > b$, $(n-1)!S_k$ is an integer, and we deduce that

$$\sum_{\nu \geq n} \frac{\sigma_k(\nu)}{(\nu)_{\nu-n+1}} \in \mathbb{N},$$

where $(x)_m = x(x-1) \cdots (x-m+1)$. Noting that for all $\varepsilon > 0$ and n sufficiently large, we have $\sigma_k(n) < n^{k+\varepsilon}$, we deduce that

$$\left\| \sum_{\nu=n}^{n+k-1} \frac{\sigma_k(\nu)}{(\nu)_{\nu-n+1}} \right\| < n^{-1+\varepsilon}.$$

Here and in the sequel, $\|x\|$ denotes the distance of x to the nearest integer. Now assume Schinzel's conjecture H, and fix some prime $p > k$. Then there are infinitely many prime numbers $q \equiv 1 \pmod{k!^k}$, such that $\frac{q+i}{i+1}$ is prime for all $i \leq k$. For such a prime number q and $i \leq k$ we have

$$\sigma_k(q+i) = \left(\left(\frac{q+i}{i+1} \right)^k + 1 \right) \sigma_k(i+1) = q^k \sigma_{-k}(i+1) + O(1),$$

hence,

$$\sum_{\nu=q}^{q+k-1} \frac{\sigma_k(\nu)}{(\nu)_{\nu-q+1}} = \sum_{i=1}^k \sigma_{-k}(i) \frac{(q+i-1)^k}{(q+i-1)_i} + O(q^{-1}).$$

The fraction $\frac{(q+i-1)^k}{(q+i-1)^i}$ can be written as $P_{k,i}(q) + O(q^{-1})$ for some polynomial $P_{k,i} \in \mathbb{Q}[x]$, combining our estimates we obtain that for all prime numbers $q \equiv 1 \pmod{k!^k}$ with $\frac{q+i}{i+1}$ prime for all $i \leq k$, we have

$$\left\| \sum_{i=1}^k \sigma_{-k}(i) P_{k,i}(q) \right\| < q^{-1+\epsilon}. \quad (1)$$

Now we repeat our argument, this time choosing an integer $q = pr$, $q \equiv 1 \pmod{k!^k}$, with r prime, such that $\frac{q+i}{i+1}$ is prime for all $i \leq k$. Arguing as above we deduce that

$$\left\| \sigma_{-k}(p) P_{k,1}(q) + \sum_{i=2}^k \sigma_{-k}(i) P_{k,i}(q) \right\| < q^{-1+\epsilon}. \quad (2)$$

Since q is fixed $\pmod{k!^k}$, the fractional part of $\sigma_{-k}(i) P_{k,i}(q)$ does not depend on q , hence, comparing (1) and (2), we deduce that

$$\| \sigma_{-k}(p) P_{k,1}(q_1) - \sigma_{-k}(1) P_{k,1}(q_2) \| < q_1^{-1+\epsilon}$$

holds true for all integers $q_1 < q_2$, such that q_1 is p times a prime, q_2 is prime, $q_1 \equiv q_2 \equiv 1 \pmod{k!^k}$, and $\frac{q_i+i}{i}$ is prime for $j = 1, 2$ and $i \leq k$. Using the fact that $P_{k,1}(x) = x^{k-1}$ and $\sigma_{-k}(1) = 1$, we obtain

$$\left\| \frac{q_1^{k-1}}{p^k} \right\| < q_1^{-1+\epsilon}.$$

For $q_1 > p^2$, the left hand side cannot vanish, since then $p^2 \nmid q_1$. Hence, the left hand side is a nonzero rational number with denominator dividing p^k , and therefore bounded below by p^{-k} . However, p is fixed, whereas q_1 may be chosen arbitrary large, which yields a contradiction.

The proof of the second statement is similar, however, due to the fact that we do not even know whether there is an infinitude of Sophie Germain primes, it becomes more technical. As a substitute for conjecture H we will use the following result. Denote by $P^-(n)$ the least prime factor of n .

Lemma. *The number of primes $p \leq x$ such that $P^-\left(\frac{p+1}{2}\right)$ and $P^-\left(\frac{p+2}{3}\right)$ are both greater than $x^{1/9}$ is $\gg \frac{x}{\log^3 x}$.*

Proof. This follows from [6, Theorem 7.4]. \square

Note that the exponent $1/9$ is not optimal, however, it is sufficient for our purpose. In the sequel, let q be a prime number satisfying $P^-\left(\frac{q+1}{2}\right) > q^{1/9}$ and $P^-\left(\frac{q+2}{3}\right) > q^{1/9}$, and suppose that q is sufficiently large. As in the proof of the first part of our theorem, we deduce that

$$\left\| \frac{\sigma_3(q)}{q} + \frac{\sigma_3(q+1)}{q(q+1)} + \frac{\sigma_3(q+2)}{q(q+1)(q+2)} \right\| < q^{-1+\epsilon}.$$

By assumption we have $\sigma_3(q+2) = q^3 + \frac{q^3}{27} + O(q^{8/3})$, that is, $\frac{\sigma_3(q+2)}{q(q+1)(q+2)} = \frac{28}{27} + O(q^{-1/3})$. Moreover, denoting by $\{x\}$ the fractional part of the real number x , we have $\left\{ \frac{\sigma_3(q)}{q} \right\} = \frac{1}{q}$, and we have

$$\left\{ \frac{\sigma_3(q+1)}{q(q+1)} - \frac{\sigma_3(q+1)}{(q+1)^2} \right\} = 1 - \frac{1}{8} + O\left(\left\| \frac{(q+1)^2}{q} \right\|\right) + O(q^{-1/3}) = \frac{7}{8} + O(q^{-1/3}).$$

Hence, setting $n = \frac{q+1}{2}$, we find that there are $\gg \frac{x}{\log^3 x}$ integers $n \leq x$ with the following properties:

(i) We have

$$\left\| \frac{9\sigma_3(n)}{4n^2} + \frac{19}{216} \right\| \ll n^{-1/3},$$

(ii) $P^-(n) > n^{-1/9}$,

(iii) $2n-1$ is prime, and $P^-\left(\frac{2n+1}{3}\right) > n^{-1/9}$.

We will obtain a contradiction by estimating the number of integers n with these properties from above. If there were as many integers n with these properties, there has to be some $k \leq 9$, such that there are $\gg \frac{x}{\log^3 x}$ integers n with these properties which have precisely k prime factors. We may assume that n is squarefree, for otherwise n was divisible by the square of an integer $k \geq n^{1/9}$, and the number of integers $n \in [x, 2x]$ with this property is bounded above by

$$\sum_{k \geq x^{1/9}} \left[\frac{2x}{k^2} \right] \ll x^{8/9},$$

which is of negligible size. Let $p_1 < p_2 < \dots < p_k$ be the prime factors of n . Set $[k] = \{1, \dots, k\}$. Then divisors of n correspond to subsets I of $[k]$, and inserting the definition of σ_3 , we see that condition (i) is equivalent to

$$\left\| \sum_{I \subseteq [k]} \frac{9 \prod_{i \in I} p_i}{4 \prod_{i \notin I} p_i^2} + \frac{19}{216} \right\| \ll n^{-1/3}.$$

The summand $I = [k]$ corresponds to the trivial divisor n , which contributes $\frac{9n}{4}$. Since for n sufficiently large, n has to be odd by condition (ii), the contribution is $\pm \frac{1}{4} \pmod{1}$. Hence, all integers satisfying (i) and (ii) also satisfy

$$\left\| \sum_{I \subseteq [k]} \frac{9 \prod_{i \in I} p_i}{4 \prod_{i \notin I} p_i^2} + \frac{19}{216} \pm \frac{1}{4} \right\| \ll n^{-1/3}, \quad (3)$$

If $k = 1$, then $n = p_1$, and (3) becomes $\left\| \frac{9}{4p_1^2} + \frac{19}{216} \pm \frac{1}{4} \right\| \ll p_1^{1/3}$, which is impossible for n sufficiently large. If $k = 2$, (3) is equivalent to

$$\left\| \frac{9p_2}{4p_1^2} + \frac{19}{216} \pm \frac{1}{4} \right\| \ll (p_1 p_2)^{-1/9}$$

since $p_2 > p_1 > n^{1/9}$. For fixed p_1 , all admissible $p_2 < x$ are contained in $\ll \frac{x}{p_1^3} + 1$ intervals of length $\ll p_1^{2-2/9}$ each, hence, the number of admissible p_2 is $\ll x p_1^{-1-2/9}$. Summing over all $p_1 > x^{1/9}$, we find that the number of integers $n \leq x$ with two prime factors satisfying (3) is bounded above by $x^{1-2/81}$. Hence, we may assume that $k \geq 3$, in particular, we have $p_1 < x^{1/3}$. We divide the interval $[x^{1/9}, x^{1/3}]$ into $\ll \log x$ intervals of the form $[y, 2y]$ and will now estimate the number of integers $n \leq x$ satisfying conditions (i)–(iii) together with $p_1 \in [y, 2y]$. Set

$$\alpha = \sum_{1 \notin I \subseteq [k]} \frac{9 \prod_{i \in I} p_i}{4 \prod_{i \notin I} p_i^2}.$$

Note that our assumption implies $p_1^2 < \alpha < x$. We now distinguish two cases, depending on the relative size of α and y . Let C be a constant to be determined later, and assume first that for each integer $2 \leq \ell \leq 9$ we have

$$\alpha \notin [y^\ell \log^{-C} x, y^\ell \log^C x]. \quad (4)$$

Then we rewrite (3) as

$$\|\alpha p_1 + \frac{\alpha}{p_1^2} + \frac{19}{216} \pm \frac{1}{4}\| \ll n^{-1/3}.$$

It suffices to show that the number of integers $n_1 \in [y, 2y]$ satisfying

$$\|\alpha n_1 + \frac{\alpha}{n_1^2} + \frac{19}{216}\| \ll x^{-1/4} \quad (5)$$

is bounded above by $\frac{y}{\log^6 y}$. This quantity is at most $yx^{-1/4} + D$, where $D = D(\alpha, y)$ is the discrepancy of the sequence $(\alpha n_1 + \frac{\alpha}{n_1^2})_{n_1 \in [y, 2y]}$. Bounding the discrepancy using the Erdős-Turán-inequality (see e.g. [7, Corollary 1.1]) we obtain

$$D \ll \frac{y}{H} + \sum_{h \leq H} \frac{1}{h} \left| \sum_{n=y}^{2y} e(hf(n)) \right|.$$

for any parameter $H \geq 1$, where we have set $f(n) = \alpha n + \frac{\alpha}{n^2}$. To bound the exponential sum on the right hand side, it suffices to use the simplest van der Corput-type estimates (see e.g. [4, Theorem 2.9]). If the integer $2 \leq \ell \leq 8$ is determined by means of the inequality $y^\ell \log^C x < \alpha < y^{\ell+1} \log^{-C} x$, we have

$$\frac{\log^C}{y} \ll f^{(\ell+1)}(x) \ll \frac{1}{\log^C x}, \quad \forall x \in [y, 2y].$$

For $\ell \geq 3$ we deduce

$$\begin{aligned} \sum_{n=y}^{2y} e(hf(n)) &\ll y(h\alpha f^{(\ell+1)}(y))^{1/(4Q-2)} + h^{-1} f^{(\ell+1)}(y)^{-1} \\ &\ll hy \log^{-C/Q} x + y \log^{-C} x, \end{aligned}$$

where $Q = 2^{\ell+1}$, and therefore

$$\begin{aligned} D &\ll \frac{y}{H} + \sum_{h \leq H} \frac{1}{h} \left| \sum_{n=y}^{2y} e(hf(n)) \right| \\ &\ll \frac{y}{H} + Hy \log^{-C/Q} x. \end{aligned}$$

Setting $H = \log^7 x$ and $C = 14Q \leq 2^{13}$, we obtain $D \ll \frac{y}{\log^7 x}$, and therefore, for x sufficiently large, $D \leq \frac{y}{\log^6 y}$. Note that, apart from (4), this estimate is independent of α , which shows that there are $\ll \frac{x}{\log^5 x}$ integers $n \leq x$ satisfying conditions (i)–(iii) together with (4).

Now we consider the case

$$\alpha \in [y^\ell \log^{-C} x, y^\ell \log^C x] \quad (6)$$

for some integer $2 \leq \ell \leq 9$. Fix prime numbers $x^{1/9} < p_2 < \dots < p_k$, and a real number y such that $yp_2 \dots p_k < x$, such that (6) is satisfied. The prime numbers p_2, \dots, p_k can be chosen in $\ll \frac{x}{y \log x}$ ways, and there are $\ll \log \log x$ intervals of the form $[y, 2y]$ to be considered. For each fixed p_2, \dots, p_k , the number of primes $p_1 \in [y, 2y]$ such that $p_1 \dots p_k$ satisfies condition (iii) is $\ll \frac{y}{\log^3 x}$, thus, the total number of integers n satisfying conditions (ii) and (iii) as well as

$$p_1^\ell \log^{-C} x \leq \alpha \leq 2p_1^\ell \log^C x$$

for some integer ℓ is $\ll \frac{x \log \log x}{\log^4 x}$. Hence, the total number of integers $n \leq x$ satisfying conditions (i)–(iii) is bounded above by $\mathcal{O}\left(\frac{x \log \log x}{\log^4 x}\right)$, which contradicts our lower bound $\frac{x}{\log^3 x}$, proving our theorem. \square

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